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FEEDBACK CONTROL OF VIBRATIONS IN A MICROMACHINED CANTILEVER BEAM WITH ELECTROSTATIC ACTUATORS

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The problem of feedback control of vibrations in a micromachined cantilever beam with nonlinear electrostatic actuators is considered. Various forms of nonlinear feedback controls depending on localized spatial averages of the beam velocity and displacement near the beam tip are derived by considering the time rate-of-change of the total energy of the beam. The physical implementation of the derived feedback controls is discussed briefly. The dynamic behaviour of the beam with the derived feedback controls is determined by computer simulation.

1. INTRODUCTION

Recent advances in VLSI technology have led to the fabrication of various types of micro-electromechanical structures [1–5]. These structures constitute integral parts of micro-sensors and micro-actuators. A common micromachined structure is the cantilever beam which is a basic part of many devices such as micro-accelerometers, micro-optical electromechanical devices, and atomic-force microscopes. In these devices, it is of importance to damp out the beam vibrations quickly and to control the position of the beam-tip precisely.

In this paper, we consider the problem of feedback control of vibrations in a micromachined cantilever beam with nonlinear electrostatic actuators. We begin with the development of suitable mathematical models for micromachined cantilever beams with electrostatic actuators. Then, various forms of feedback controls for damping the vibrations of the beam are derived. The physical implementation of these feedback controls is discussed briefly. Finally, the dynamic behaviour of the beam with the derived feedback controls is determined by computer simulation.

2. MATHEMATICAL MODELS

Figure 1 shows a sketch of a cantilever beam with rectangular cross-section, which is micromachined from polysilicon or other material. The free-end of the beam is coated with a thin electrically conducting film which is at the ground potential. To control the beam motion, appropriate voltages are applied to the actuators which are formed by two fixed plane electrodes located near the end of the beam. Since the electrostatic force between a plate and the beam tip is always attractive, it is necessary to use one actuator to produce force in one direction, and another actuator in the opposite direction.

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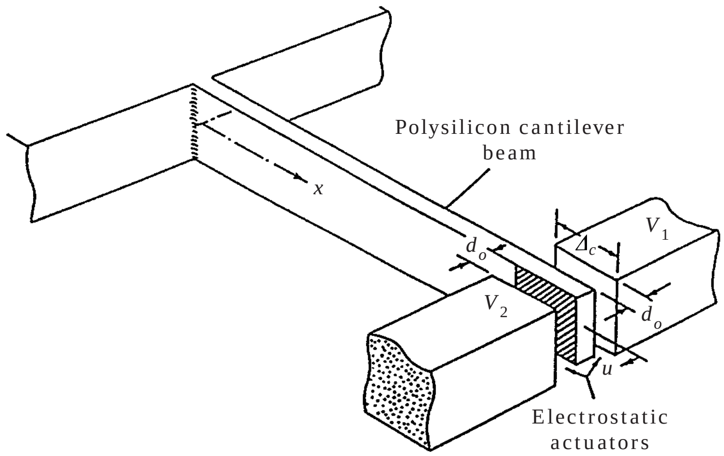

Figure 1. Sketch of a micromachined cantilever beam with electrostatic actuators.

We assume that the undeformed cantilever beam is slender and straight, with length $L + \Delta_c/2$ and width W. Moreover, the deformation is small so that linear theory of elastic beams is applicable. Thus, the beam displacement u is describable by

$$\rho u_{tt}(t, x) = -(EIu_{xx})_{xx}(t, x) + F_e(x, u(t, x)), \qquad t > 0, \qquad x \in \Omega(\Delta_c) \stackrel{\text{def}}{=}]0, L + \Delta_c/2[, \tag{1}$$

with boundary conditions:

$$u(t, 0) = 0, \qquad u_x(t, 0) = 0, \qquad (EIu_{xx})(t, L + \Delta_c/2) = 0, \qquad (EIu_{xx})_x(t, L + \Delta_c/2) = 0, \tag{2}$$

along with initial data and constraint:

$$u(0, x) = u_0(x), \qquad u_t(0, x) = \hat{u}_0(x), \qquad x \in \Omega(\Delta_c); \qquad \max_{x \in \Omega_a(\Delta_c)} |u(t, x)| \leqslant \hat{d}_0, \tag{3}$$

where the subscripts x and t denote partial differentiation with respect to the corresponding variables; u_0 and $\hat{u}_0$ are specified functions of x; $\Omega_a(\Delta_c) \stackrel{\text{def}}{=} [L - \Delta_c/2, L + \Delta_c/2]$ is the effective spatial domain of the actuators; ρ and EI denote the linear mass density and bending stiffness respectively.

The electrostatic force per unit length due to the actuators is given by

$$F_e(x, u(t, x)) = \frac{\varepsilon_0 W}{2} \left\{ \frac{V_2^2}{(d_0 - u(t, x))^2} - \frac{V_1^2}{(d_0 + u(t, x))^2} \right\} \phi(x), \tag{4}$$

where ϕ is the characteristic function of the actuator's effective spatial domain $\Omega_a(\Delta_c)$, or a non-negative spatial weighting function having compact support in $\Omega_a(\Delta_c)$; ε_0 and d_0 denote the permittivity of free space and the actuator gap respectively. The parameter $\hat{d}_0$ in equation (3) is set at a suitable value less than d_0 so that the elastic restoring force always exceeds the actuator's electrostatic force. Since the electrostatic forces due to V_1^2 and V_2^2 are attractive, the controls are *unilateral*. Evidently, only *one* control V_1 or V_2 should be active at any time, otherwise the controls would be working against each other. From the fact that the beam displacement u is of the same order of magnitude as that of the actuator gap d_0, it is essential that we retain the nonlinear model for F_e given by equation (4).

Let $u_d = u_d(x)$ be the desired static deformation satisfying the equilibrium equation corresponding to equations (1), (2) and (4) and constraint in equation (3) with specified constant actuator voltages V_{1d} or V_{2d}. We assume that for $u_d(L) > 0$ (resp. <0), only V_{2d} (resp. V_{1d}) is active. That is, V_{1d} (resp. V_{2d}) is zero for $u_d(L) > 0$ (resp. <0), and $V_{1d} = V_{2d} = 0$ for $u_d(L) = 0$. Moreover, we assume that $\max_{x\in\Omega_a(\Delta_c)} |u_d(x)|$ is sufficiently small so that the equilibrium deformation is statically stable. Thus,

$$(EIu_{dxx})_{xx}(x) = \begin{cases} \dfrac{\varepsilon_0 W V_{2d}^2 \phi(x)}{2(d_0 - u_d(x))^2}, & \text{if } u_d(L) > 0; \\ 0, & \text{if } u_d(L) = 0; \\ \dfrac{-\varepsilon_0 W V_{1d}^2 \phi(x)}{2(d_0 + u_d(x))^2}, & \text{if } u_d(L) < 0. \end{cases} \tag{5}$$

Let $\delta u = u_d - u$. Then, δu satisfies

$$\rho \delta u_{tt}(t, x) = -(EI\delta u_{xx})_{xx}(t, x) + f_e(u_d(x), \delta u(t, x))\phi(x); \qquad t > 0, \qquad x \in \Omega(\Delta_c), \tag{6}$$

with boundary conditions:

$$\delta u(t, 0) = 0, \qquad \delta u_x(t, 0) = 0, \qquad (EI\delta u_{xx})(t, L + \Delta_c/2) = 0,$$
$$(EI\delta u_{xx})_x(t, L + \Delta_c/2) = 0, \tag{7}$$

and initial data:

$$\delta u(0, x) = u_d(x) - u_0(x), \qquad \delta u_t(0, x) = -\hat{u}_0(x), \qquad x \in \Omega(\Delta_c), \tag{8}$$

where

$$f_e(u_d(x), \delta u(t, x))$$
$$= \frac{\varepsilon_0 W}{2} \begin{cases} \dfrac{V_{2d}^2}{(d_0 - u_d(x))^2} - \dfrac{V_2^2}{(d_0 - u_d(x) + \delta u(t, x))^2} + \dfrac{V_1^2}{(d_0 + u_d(x) - \delta u(t, x))^2}, & \text{if } u_d(L) > 0; \\ \dfrac{-V_2^2}{(d_0 + \delta u(t, x))^2} + \dfrac{V_1^2}{(d_0 - \delta u(t, x))^2}, & \text{if } u_d(L) = 0; \\ \dfrac{-V_{1d}^2}{(d_0 + u_d(x))^2} - \dfrac{V_2^2}{(d_0 - u_d(x) + \delta u(t, x))^2} + \dfrac{V_1^2}{(d_0 + u_d(x) - \delta u(t, x))^2}, & \text{if } u_d(L) < 0. \end{cases} \tag{9}$$

In the case where $L \gg \Delta_c$ so that the distributed electrostatic actuator forces can be approximated by forces concentrated at the beam tip, we have the following simplified model:

$$\rho \delta u_{tt}(t, x) = -(EI\delta u_{xx})_{xx}(t, x), \qquad t > 0, \qquad x \in \Omega(0) =]0, L[, \tag{10}$$

along with initial data given in equation (3) defined for $x \in \Omega(0)$, and constraint $|u_d(t, L) - \delta u(t, L)| \leqslant \hat{d}_0$. The boundary conditions are

$$\delta u(t, 0) = 0, \qquad \delta u_x(t, 0) = 0, \qquad (EI\delta u_{xx})(t, L) = 0, \tag{11}$$

$$-(EI\delta u_{xx})_x(t, L) = \Delta_c f_e(u_d(L), \delta u(t, L)), \tag{12}$$

where f_e is defined in equation (9).

3. STABILIZING FEEDBACK CONTROLS

The objective is to derive explicit expressions for the feedback controls V_1^2 and V_2^2 for damping the beam vibrations. The approach taken here is to consider the total energy corresponding to the perturbed beam equation (6) at any time t given by

$$\mathscr{E}(t) = \frac{1}{2}\int_0^{L+\Delta_c/2} (\rho|\delta u_t(t,x)|^2 + EI|\delta u_{xx}(t,x)|^2)\,\mathrm{d}x. \tag{13}$$

The time rate-of-change of $\mathscr{E}$, in view of equations (6), (7) and (9), is given by

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = \int_{L-\Delta_c/2}^{L+\Delta_c/2} \delta u_t(t,x) f_e(u_d(x), \delta u(t,x))\phi(x)\,\mathrm{d}x$$

$$= \frac{\varepsilon_0 W}{2}\begin{cases} \alpha_1^+(t,u_d)V_{2d}^2 - \alpha_2(t,u_d)V_2^2 + \alpha_3(t,u_d)V_1^2, & \text{if } u_d(L) > 0; \\ -\alpha_2(t,0)V_2^2 + \alpha_3(t,0)V_1^2, & \text{if } u_d(L) = 0; \\ -\alpha_1^-(t,u_d)V_{1d}^2 - \alpha_2(t,u_d)V_2^2 + \alpha_3(t,u_d)V_1^2, & \text{if } u_d(L) < 0, \end{cases} \tag{14}$$

where

$$\alpha_1^+(t,u_d) = \int_{L-\Delta_c/2}^{L+\Delta_c/2} \frac{\delta u_t(t,x)\phi(x)\,\mathrm{d}x}{(d_0 - u_d(x))^2}, \qquad \alpha_1^-(t,u_d) = \int_{L-\Delta_c/2}^{L+\Delta_c/2} \frac{\delta u_t(t,x)\phi(x)\,\mathrm{d}x}{(d_0 + u_d(x))^2},$$

$$\alpha_2(t,u_d) = \int_{L-\Delta_c/2}^{L+\Delta_c/2} \frac{\delta u_t(t,x)\phi(x)\,\mathrm{d}x}{(d_0 - u_d(x) + \delta u(t,x))^2},$$

$$\alpha_3(t,u_d) = \int_{L-\Delta_c/2}^{L+\Delta_c/2} \frac{\delta u_t(t,x)\phi(x)\,\mathrm{d}x}{(d_0 + u_d(x) - \delta u(t,x))^2}. \tag{15}$$

We wish to derive stabilizing feedback controls V_1^2 and V_2^2 such that the resulting feedback-controlled beam is *dissipative* in the sense that $\mathrm{d}\mathscr{E}(t)/\mathrm{d}t \leqslant 0$ for all $t \geqslant 0$.

First, we consider the simplest case where $u_d(L) = 0$, which can be attained by setting $V_{1d} = V_{2d} = 0$. Since both V_1^2 and V_2^2 are nonnegative, it is evident from equation (14) that $V_1^2(t) = 0$ if $\alpha_3(t,0) > 0$, and $V_2^2(t) = 0$ if $\alpha_2(t,0) < 0$. When $\alpha_3(t,0) < 0$ (resp. $\alpha_2(t,0) > 0$), a possible choice is $V_1^2(t) = K_1|\alpha_3(t,0)|$ with $K_1 > 0$ (resp. $V_2^2(t) = K_2\alpha_2(t,0)$ with $K_2 > 0$). The foregoing choices lead to the following stabilizing feedback controls:

$$V_1^2(t) = K_1(|\alpha_3(t,0)| - \alpha_3(t,0))/2, \qquad V_2^2(t) = K_2(|\alpha_2(t,0)| + \alpha_2(t,0))/2, \tag{16}$$

which result in

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = \frac{\varepsilon_0 W}{4}\{-K_2(|\alpha_2(t,0)|\alpha_2(t,0) + \alpha_2^2(t,0)) + K_1(|\alpha_3(t,0)|\alpha_3(t,0) - \alpha_3^2(t,0))\} \leqslant 0. \tag{17}$$

Next, we consider the case where $u_d(L) > 0$. From equation (14), we have

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = \frac{\varepsilon_0 W}{2}(\alpha_1^+(t, u_d)V_{2d}^2 - \alpha_2(t, u_d)V_2^2 + \alpha_3(t, u_d)V_1^2). \tag{18}$$

Setting

$$V_1^2(t) = \frac{K_1}{2}(|\alpha_3(t, u_d)| - \alpha_3(t, u_d)), \qquad K_1 > 0, \tag{19a}$$

and

$$V_2^2(t) = \begin{cases} \dfrac{K_2}{2}(|\alpha_2(t, u_d)| + \alpha_2(t, u_d)) + \dfrac{\alpha_1^+(t, u_d)}{\alpha_2(t, u_d)} V_{2d}^2, & \text{if } \alpha_2(t, u_d) \neq 0; \\ 0, & \text{if } \alpha_2(t, u_d) = 0 \end{cases}, \qquad K_2 > 0, \tag{19b}$$

in equation (18) gives

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = -\frac{\varepsilon_0 W}{4}\{K_2\alpha_2(t, u_d)(|\alpha_2(t, u_d)| + \alpha_2(t, u_d))$$
$$-K_1(|\alpha_3(t, u_d)| - \alpha_3(t, u_d))\alpha_3(t, u_d)\} < 0, \qquad \text{if } \alpha_2(t, u_d) \neq 0. \tag{20}$$

If both $\alpha_2(t, u_d)$ and $\alpha_3(t, u_d)$ are equal to zero, then the controls V_1^2 and V_2^2 have no effect on $\mathrm{d}\mathscr{E}(t)/\mathrm{d}t$. If in addition, $\alpha_1^+(t, u_d) > 0$, then $\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = \varepsilon_0 W\alpha_1^+(t, u_d)V_{2d}^2/2 > 0$. We note from equation (19b) that $V_2^2(t) \geqslant 0$ if $\alpha_2^+(t, u_d)$ and $\alpha_2(t, u_d)$ have the same sign. It is evident from equation (15) that the foregoing condition holds if $\delta u_t(t, u_d)$ does not change sign over $\Omega_a(\Delta_c)$. In a real beam, Δ_c is usually small compared to the beam length L. Moreover, $\delta u_t(t, x)$ does not change sign over $\Omega_a(\Delta_c)$. Consequently, all the α coefficients defined in equation (15) have the same sign as that of $\delta u_t(t, x)$, and are equal to zero if and only if $\delta u_t(t, x) \equiv 0$ in $\Omega_a(\Delta_c)$. Thus, the situation that $\alpha_1^+(t, u_d) > 0$, $\alpha_2(t, u_d) = \alpha_3(t, u_d) = 0$ does not occur in a real beam.

Similarly, for the case where $u_d(L) < 0$, we set

$$V_1^2(t) = \begin{cases} \dfrac{K_1}{2}(|\alpha_3(t, u_d)| - \alpha_3(t, u_d)) + \dfrac{\alpha_1^-(t, u_d)}{\alpha_3(t, u_d)} V_{1d}^2, & \text{if } \alpha_3(t, u_d) \neq 0; \\ 0, & \text{if } \alpha_3(t, u_d) = 0 \end{cases}, \qquad K_1 > 0, \tag{21a}$$

and

$$V_2^2(t) = \frac{K_2}{2}(|\alpha_2(t, u_d)| + \alpha_2(t, u_d)), \qquad K_2 > 0. \tag{21b}$$

Then, we have

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = -\frac{\varepsilon_0 W}{4}\{K_2\alpha_2(t, u_d)(|\alpha_2(t, u_d)| + \alpha_2(t, u_d))$$
$$- K_1(|\alpha_3(t, u_d)| - \alpha_3(t, u_d))\alpha_3(t, u_d)\} < 0, \qquad \text{if } \alpha_3(t, u_d) \neq 0. \tag{22}$$

The foregoing feedback controls involve weighted averages of the beam velocity over the actuator domain which are not readily implementable physically. In what follows, we shall focus our attention on the important special case where $L \gg \Delta_c$ so that the beam is describable by equations (10), (11) and (12). The desired static deformation u_d is described by

$$(EIu_{dxx})_{xx}(x) = 0, \qquad x \in \Omega(0) =]0, L[, \tag{23}$$

with boundary conditions:

$$u_d(0) = 0, \qquad u_{dx}(0) = 0, \qquad (EIu_{dxx})(L) = 0,$$

$$-(EIu_{dxx})_x(L) = \frac{\varepsilon_0 \Delta_c W}{2} \begin{cases} \dfrac{V_{2d}^2}{(d_0 - u_d(L))^2}, & \text{if } u_d(L) > 0; \\ 0, & \text{if } u_d(L) = 0; \\ \dfrac{-V_{1d}^2}{(d_0 + u_d(L))^2}, & \text{if } u_d(L) < 0. \end{cases} \tag{24}$$

Now, we consider the total energy $\mathscr{E}$ corresponding to the perturbed beam equation (10). Here, $\mathscr{E}(t)$ is given by equation (13) with Δ_c set to zero. The time rate-of-change of $\mathscr{E}$, in view of equation (10) and boundary conditions (11) and (12), is given by

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = \frac{\varepsilon_0 \Delta_c W}{2} \delta u_t(t,L)$$

$$\begin{cases} \dfrac{V_{2d}^2}{(d_0 - u_d(L))^2} - \dfrac{V_2^2}{(d_0 - u_d(L) + \delta u(t, L))^2} + \dfrac{V_1^2}{(d_0 + u_d(L) - \delta u(t, L))^2}, & \text{if } u_d(L) > 0; \\ \dfrac{-V_2^2}{(d_0 + \delta u(t, L))^2} + \dfrac{V_1^2}{(d_0 - \delta u(t, L))^2}, & \text{if } u_d(L) = 0; \\ \dfrac{-V_{1d}^2}{(d_0 + u_d(L))^2} - \dfrac{V_2^2}{(d_0 - u_d(L) + \delta u(t, L))^2} + \dfrac{V_1^2}{(d_0 + u_d(L) - \delta u(t, L))^2}, & \text{if } u_d(L) < 0. \end{cases} \tag{25}$$

First, we consider the case where $u_d(L) = 0$. Simple stabilizing feedback controls can be obtained by exact linearization. If $\delta u_t(t, L) > 0$, set $V_1^2(t) \equiv 0$ and $V_2^2(t) = K\delta u_t(t, L)(d_0 + \delta u(t, L))^2$; and if $\delta u_t(t, L) < 0$, set $V_2^2(t) \equiv 0$ and $V_1^2(t) = -K\delta u_t(t, L)(d_0 - \delta u(t, L))^2$, or

$$V_1^2(t) = -K\delta u_t(t, L)(d_0 - \delta u(t, L))^2(1 - \operatorname{sgn}(\delta u_t(t, L)))/2,$$
$$V_2^2(t) = K\delta u_t(t, L)(d_0 + \delta u(t, L))^2(1 + \operatorname{sgn}(\delta u_t(t, L)))/2, \tag{26}$$

where K is a positive constant; $\operatorname{sgn}(h) = h/|h|$ if $|h| \neq 0$, and $\operatorname{sgn}(0) = 0$. With feedback controls given by equation (26), we have

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = -\frac{\varepsilon_0 \Delta_c WK}{2} |\delta u_t(t, L)|^2 \leqslant 0. \tag{27}$$

Moreover, the nonlinear boundary condition (12) becomes a linear one given by

$$(EI\delta u_{xx})_x(t, L) = \frac{\varepsilon_0 \Delta_c WK}{2} \delta u_t(t, L). \tag{28}$$

Consequently, the resulting feedback-controlled beam, in the absence of beam-tip displacement constraint, is a linear system which is known to be exponentially stable in the sense that $\mathscr{E}(t)$ decays exponentially with time t [6]. Unfortunately, this seemingly simple solution to the feedback stabilization problem is unsatisfactory, since V_1^2 (resp. V_2^2)$\to 0$ as $u(t, L) \to -d_0$ (resp. d_0), which implies that, as the magnitude of the beam-tip displacement approaches d_0, the magnitude of the active restoring control force decreases. Moreover, very large values of feedback gain K are necessary to produce significant actuator voltages. To avoid the abovementioned undesirable features, we modify the feedback controls (26) as follows:

$$V_1^2(t) = -K\delta u_t(t, L)(1 - \operatorname{sgn}(\delta u_t(t, L)))/2, \qquad V_2^2(t) = K\delta u_t(t, L)(1 + \operatorname{sgn}(\delta u_t(t, L)))/2. \tag{29}$$

In this case, we have

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = -\frac{\varepsilon_0 \Delta_c WK|\delta u_t(t, L)|^2}{2(d_0 + \delta u(t, L)\operatorname{sgn}(\delta u_t(t, L)))^2} \leqslant 0, \tag{30}$$

and boundary condition (12) takes on the form:

$$(EI\delta u_{xx})_x(t, L) = \frac{\varepsilon_0 \Delta_c WK}{2} g(\delta u(t, L), \delta u_t(t, L))\delta u_t(t, L), \tag{31a}$$

where

$$g(\delta u(t, L), \delta u_t(t, L)) = \frac{1}{(d_0 + \delta u(t, L)\operatorname{sgn}(\delta u_t(t, L)))^2}. \tag{31b}$$

We note that equation (31) is similar to equation (28) except for the term g which depends nonlinearly on $\delta u(t, L)$ and $\delta u_t(t, L)$. Since for $|\delta u(t, L)| < d_0$ and $g(\delta u(t, L), \delta u_t(t, L)) > 0$, we expect that the feedback-controlled beam behaves somewhat like the one with linear boundary condition (28).

When the controls satisfy the magnitude constraints

$$0 \leqslant V_i^2(t) \leqslant \bar{V}^2 < \infty, \qquad i = 1, 2, \tag{32}$$

the stabilizing feedback controls become

$$V_1^2(t) = \bar{V}^2(1 - \operatorname{sgn}(\delta u_t(t, L)))/2, \qquad V_2^2(t) = \bar{V}^2(1 + \operatorname{sgn}(\delta u_t(t, L)))/2. \tag{33}$$

In this case, the boundary condition (12) takes on the form:

$$(EI\delta u_{xx})_x(t, L) = \frac{\varepsilon_0 \Delta_c W\bar{V}^2}{2}\operatorname{sgn}(\delta u_t(t, L)), \tag{34}$$

and

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = -\frac{\varepsilon_0 \Delta_c W\bar{V}^2}{2(d_0 + \delta u(t, L)\operatorname{sgn}(\delta u_t(t, L)))^2} \leqslant 0. \tag{35}$$

We observe that equation (34) can be obtained from equation (28) by replacing $K\delta u_t(t, L)$ by $\bar{V}^2\operatorname{sgn}(\delta u_t(t, L))$.

Next, we consider the case where $u_d(L) > 0$. From equation (25), it is clear that when $\delta u_t(t, L) > 0$, $V_1^2(t)$ should be set to zero. If we set

$$V_2^2(t) = \frac{1}{2}\left\{K_2 + \frac{V_{2d}^2}{(d_0 - u_d(L))^2}\right\}(d_0 - u_d(L) + \delta u(t, L))^2(1 + \operatorname{sgn}(\delta u_t(t, L))), \tag{36}$$

with $K_2 > 0$, then $V_2^2(t) \geqslant 0$, and

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = -\frac{\varepsilon_0 \Delta_c WK_2}{2}\delta u_t(t, L) \leqslant 0. \tag{37}$$

When $\delta u_t(t, L) < 0$, V_2^2 should be set to zero. If we set

$$V_1^2(t) = \frac{1}{2}\left\{K_1 - \frac{V_{2d}^2}{(d_0 - u_d(L))^2}\right\}(d_0 + u_d(L) - \delta u(t, L))^2(1 - \operatorname{sgn}(\delta u_t(t, L))), \tag{38}$$

where K_1 satisfies

$$K_1 > \frac{V_{2d}^2}{(d_0 - u_d(L))^2}, \tag{39}$$

then

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t = -\frac{\varepsilon_0 \Delta_c W K_1}{2}|\delta u_t(t, L)| \leqslant 0. \tag{40}$$

Thus, with feedback controls (36) and (38), boundary condition (12) becomes

$$(EI\delta u_{xx})_x(t, L) = \frac{\varepsilon_0 \Delta_c W}{4}\{K_1 - K_2 - (K_1 + K_2)\operatorname{sgn}(\delta u_t(t, L))\}, \tag{41}$$

and

$$\mathrm{d}\mathscr{E}(t)/\mathrm{d}t \leqslant -\frac{\varepsilon_0 \Delta_c W}{2}\min\{K_1, K_2\}|\delta u_t(t, L)| \leqslant 0. \tag{42}$$

When $K_1 = K_2 = K$, equation (41) reduces to

$$(EI\delta u_{xx})_x(t, L) = -\frac{\varepsilon_0 \Delta_c W K}{2}\operatorname{sgn}(\delta u_t(t, L)), \tag{43}$$

whose form is identical to that of equation (34).

Similarly, for the case where $u_d(L) < 0$, if we set

$$V_1^2(t) = \left\{K_1 + \frac{V_{1d}^2}{(d_0 + u_d(L))^2}\right\}(d_0 + u_d(L) - \delta u(t, L))^2(1 - \operatorname{sgn}(\delta u_t(t, L)))/2,$$

$$V_2^2(t) = \left\{K_2 - \frac{V_{1d}^2}{(d_0 + u_d(L))^2}\right\}(d_0 - u_d(L) + \delta u(t, L))^2(1 + \operatorname{sgn}(\delta u_t(t, L)))/2, \tag{44}$$

with $K_1 > 0$ and

$$K_2 > \frac{V_{1d}^2}{(d_0 + u_d(L))^2}, \tag{45}$$

then $\mathrm{d}\mathscr{E}(t)/\mathrm{d}t$ also satisfies equation (42).

To ensure that the foregoing feedback controls are indeed stabilizing controls for the given mathematical models, it is necessary to establish the existence of solutions to the corresponding feedback-controlled beam equation for all $t \geqslant 0$ while satisfying the beam-tip displacement magnitude constraint. This task can be accomplished by rewriting the feedback-controlled beam equation in form of a nonlinear Volterra integral equation, and making use of Schauder's fixed-point theorem. The mathematical details are given in reference [7]. Here, we note that for every feedback control derived earlier, the corresponding energy decay rate is proportional to the feedback gains K, K_1 or K_2. For the simplified model (10)–(12), it can be deduced that for any given positive values of the feedback gains, if the initial beam displacement and velocity satisfy $|u_0(x)| \leqslant \hat{d}_0$ and $|\hat{u}_0(x)| \leqslant \eta$ for all $x \in [0, L]$, then the solution of the corresponding feedback-controlled beam equation satisfies $|u(t, x)| \leqslant \hat{d}_0$ for all $x \in [0, L]$ and $t \geqslant 0$ for sufficiently small η.

To implement the feedback controls given by equations (29), (33), (36), (38), or (44), it is necessary to measure the instantaneous beam-tip displacement and velocity. Moreover, due to the small capacitances of the electrostatic actuators, it is essential to integrate the sensors and feedback controller with the actuators. A possible approach is to use the actuators also as displacement sensors by incorporating them in a capacitance bridge circuit driven by a high-frequency source. The beam-tip velocity can be estimated from past sampled values of the beam-tip displacement. We note that the derived feedback controls only require a knowledge of the actuator gap parameter d_0 which can be accurately measured. Thus, the controls are essentially model independent. Finally, to avoid possible damage to the actuators resulting from large beam-tip displacements, the active actuator voltage is set to zero when $|u(t, L)| > \hat{d}_0$ so that the so-called "pull-in" phenomenon (i.e. the electrostatic force exceeds the elastic restoring force) does not occur.

In the foregoing development, we have considered only electrostatic actuators. The results, with minor modifications, can also be applied to a micromachined cantilever beam with electromagnetic actuators. In this case, the beam tip is coated with ferromagnetic material, and the actuators consist of electromagnets with planar pole surfaces. The magnetic force density acting on the beam is given by

$$F_m(x, u(t, x)) = \frac{\kappa W}{2}\left\{\frac{I_2^2}{(d_0 - u(t, x))^2} - \frac{I_1^2}{(d_0 + u(t, x))^2}\right\}\phi(x), \tag{46}$$

where κ is a proportionality constant, and I_i is the electric current in the ith actuator coil. Evidently, with obvious change of parameters, all the results for the electrostatic actuators are also applicable.

4. SIMULATION RESULTS

Although the feedback controls derived in section 3 ensure that the feedback-controlled beam is dissipative, sharp estimates for the energy decay as a function of time are not readily obtainable. Except for the case with feedback controls given by equation (26), the equations for the feedback-controlled beam have nonlinear boundary conditions. To obtain some idea on the time-domain behaviour of the beam with the derived feedback controls, computer simulation studies are made for the simplified model for a uniform beam given by equations (10) and (11). Since the controls are located at the beam tip, it is more convenient to use finite-difference approximation for solving the feedback-controlled beam equation. Let $w_1 = u_t$ and $w_2 = au_{xx}$, where $a^2 = EI/\rho$. Then, equations (10) and (11) can be rewritten as

$$w_{1t} = -aw_{2xx}, \qquad w_{2t} = aw_{1xx}, \tag{47}$$

with boundary conditions:

$$w_1(t, 0) = 0, \qquad w_{1x}(t, 0) = 0, \qquad w_2(t, L) = 0; \tag{48a}$$

$$-w_{2x}(t, L) = \frac{\varepsilon_0 \Delta_c W}{2a\rho}\left\{\frac{V_2^2}{(d_0 - u(t, L))^2} - \frac{V_1^2}{(d_0 + u(t, L))^2}\right\}, \tag{48b}$$

where

$$u(t, L) = u(0, L) + \int_0^t w_1(\tau, L)\, \mathrm{d}\tau. \tag{49}$$

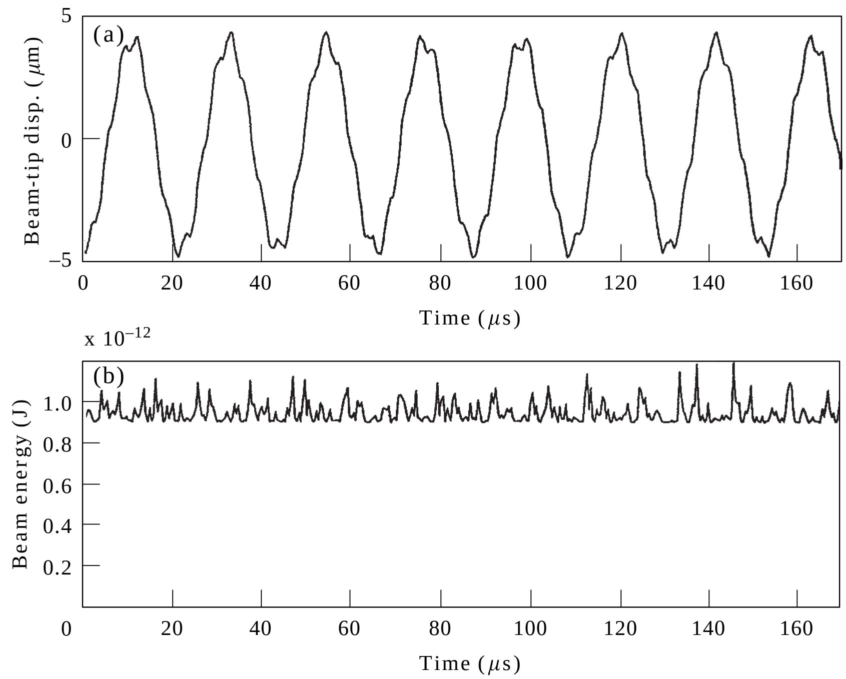

Figure 2. (a) Tip displacement and (b) total energy of uncontrolled beam.

Let Δt be the time step size, and the beam be divided into N equal segments with length $\Delta x = L/N$. Let $w_i(k, j) = w_i(k\Delta t, j\Delta x)$, $i = 1, 2$; $j = 1, \ldots, N$; $k = 0, 1, 2, \ldots$. Using forward and backward time difference for the first and second equations in equation (47) respectively leads to

$$w_1(k+1, j) = w_1(k, j) - \alpha\delta^2 w_2(k, j), \tag{50a}$$

$$w_2(k+1, j) = w_2(k, j) + \alpha\delta^2 w_1(k+1, j),$$
$$j = 1, \ldots, N; \qquad k = 0, 1, 2, \ldots, \tag{50b}$$

where $\alpha = a\Delta t/(\Delta x)^2$, and

$$\delta^2 w_i(k, j) \stackrel{\text{def}}{=} w_i(k, j+1) - 2w_i(k, j) + w_i(k, j-1), \qquad i = 1, 2. \tag{51}$$

From boundary condition (48a), we have

$$w_1(k, 1) = 0, \qquad w_2(k, N) = 0. \tag{52}$$

For $j = N$, equation (50a) becomes

$$w_1(k+1, N) = w_1(k, N) - \alpha(w_2(k, N+1) - 2w_2(k, N) + w_2(k, N-1)). \tag{53}$$

To eliminate the term $w_2(k, N+1)$ in equation (53), we make use of the following central difference approximation for boundary condition (48b):

$$\frac{w_2(k, N+1) - w_2(k, N-1)}{2(\Delta x)} = -\frac{\varepsilon_0 \Delta_c W}{2a\rho}\left\{\frac{V_2^2(k)}{(d_0 - u(k, L))^2} - \frac{V_1^2(k)}{(d_0 + u(k, L))^2}\right\}, \tag{54}$$

where $u(k, L)$ is described by the following difference equation corresponding to equation (49):

$$u(k+1, L) = u(k, L) + \Delta t w_1(k, N). \tag{55}$$

In the uncontrolled case ($V_1^2(t)$, $V_2^2(t) \equiv 0$ for all $t \geqslant 0$), it is known [7] that the foregoing difference scheme is stable if

$$\Delta t < \frac{(\Delta x)^2}{2a}, \tag{56}$$

and the truncation error is $O((\Delta t)^2) + O((\Delta x)^2)$.

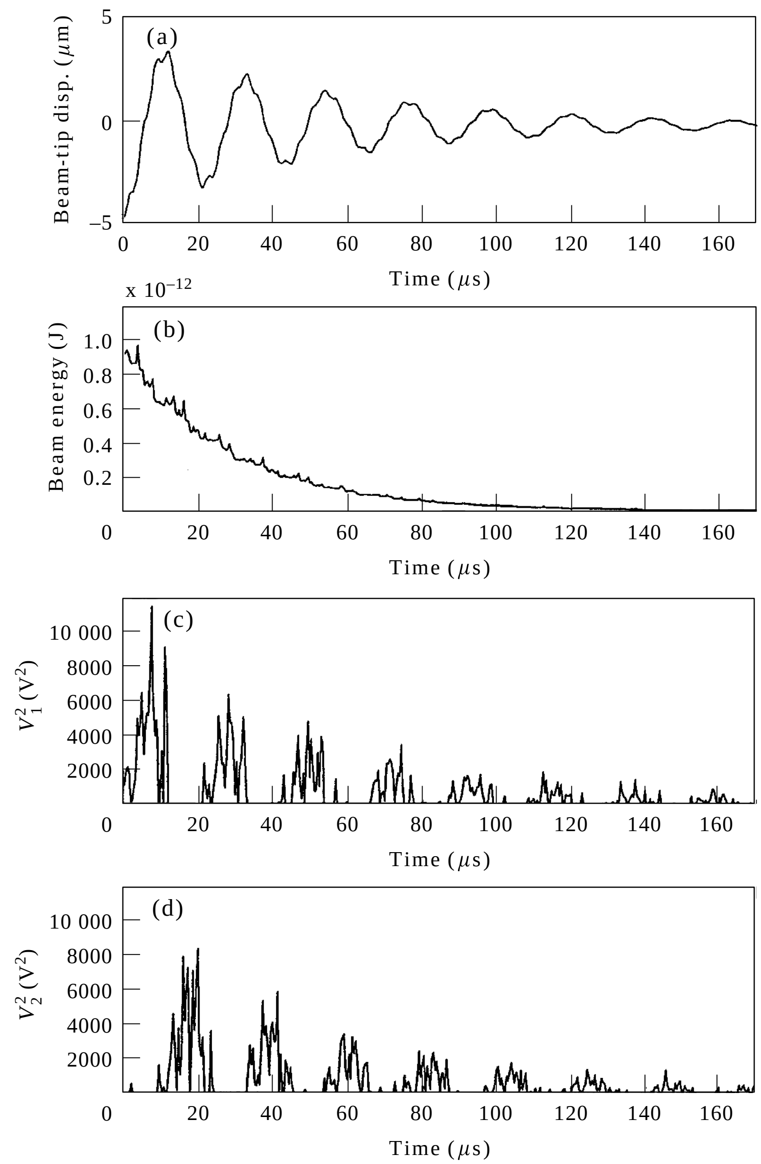

Figure 3. (a) Tip displacement, (b) total energy, and actuator voltages of beam [(c) V_1, (d) V_2] with feedback control (26) and $K = 5 \times 10^{13}$.

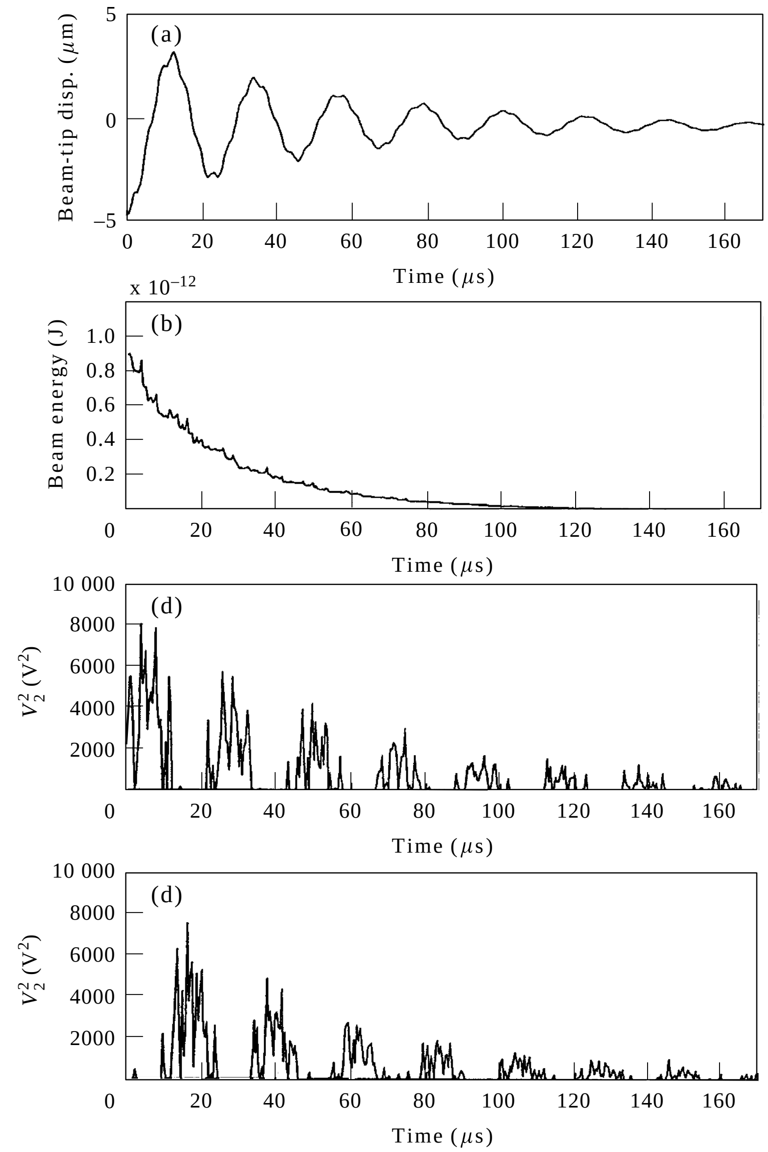

Figure 4. (a) Tip displacement, (b) total energy, and actuator voltages of beam [(c) V_1, (d) V_2] with feedback control (29) and $K = 5 \times 10^3$.

In the computer simulation study, we consider a typical micromachined cantilever beam whose parameters are given in the Appendix. We set $N = 20$ and $\Delta t = 4{\cdot}3462 \times 10^{-9}$ s to satisfy equation (56). To check how well the energy is conserved in the approximate system, numerical solutions for equations (50)–(55) are obtained for the uncontrolled case with initial data:

$$u(0, x) = -500x^2, \qquad u_t(0, x) \equiv 0, \qquad x \in [0, L]. \tag{57}$$

The results are shown in Figure 2. It can be seen that the fluctuations of the total energy of the beam due to the finite-difference approximation is within $\pm 15\%$ about the average value. Next, numerical solutions are obtained for various forms of the derived feedback controls corresponding to $u_d(L) = 0$. Figure 3 shows the beam-tip displacement, total energy, and controls as functions of time with initial data given by equations (57) and feedback controls given by equation (26) with $K = 5 \times 10^{13}$. It can be seen that both the beam-tip displacement and total energy decay toward zero as time increases. Since the controls are unilateral, the actuators are activated alternatively due to beam-tip vibrations. Note that in this case, a very high value of feedback gain K is necessary to produce effective actuation. Figures 4 and 5 show the corresponding results for feedback controls given by equation (29) with feedback gain $K = 5 \times 10^3$, and equation (33) with $\overline{V}^2 = 3 \times 10^3$ V^2. The slight offset in the steady-state beam-tip displacement from zero can be attributed to the finite-difference approximation. From Figures 3 to 5, we observe that the energy decay

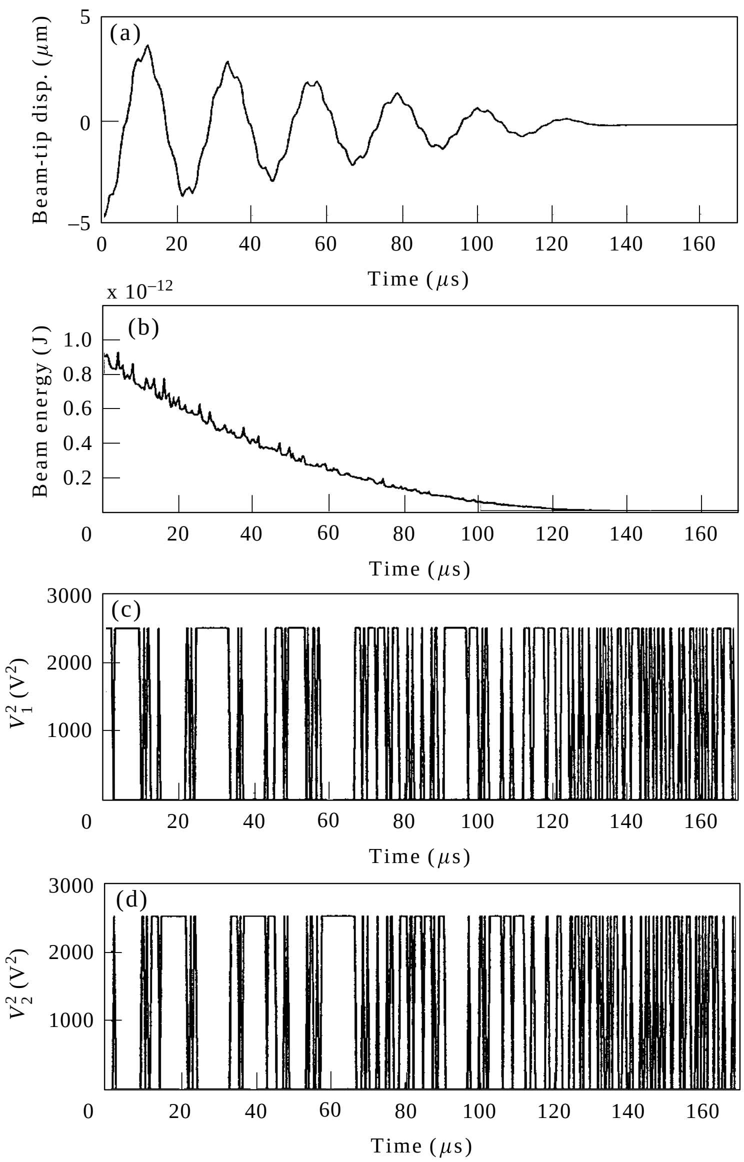

Figure 5. (a) Tip displacement, (b) total energy, and actuator voltages of beam [(c) V_1, (d) V_2] with feedback control (33) and $\overline{V}^2 = 3 \times 10^3$ V^2.

rates are essentially the same. Moreover, the magnitude of the beam-tip displacement for each case remains within the 10 μm actuator gap d_0 at all times. From the practical standpoint, feedback control (26) is not implementable physically due to the large value of feedback gain K. The chattering in control (33) can be eliminated by replacing the signum function by an appropriate saturation function.

5. CONCLUSION

The implementation of controls for micro-electromechanical systems requires integration of the controllers with the system structures. Thus, the controllers should be as simple as possible. In this study, simple nonlinear feedback controls for damping the vibrations of a micromachined cantilever beam were derived. But only certain feedback controls such as equations (29), (33), (36), (38) and (44) are suitable for physical implementation. Simulation studies for the simplified case with $u_d(L) = 0$ show that these controls are effective. The validation of these controls should be determined experimentally using real micromachined beams.

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APPENDIX: PARAMETER VALUES FOR COMPUTER SIMULATION

Beam material, SiO_2; beam dimensions (μm), L (length): 100, W (width): 25, T (thickness): 0.5; linear mass density (kg/m), $\rho = 31{\cdot}25$; cross sectional area moment of inertia (m^4), $I = WT^3/12 = 2{\cdot}604 \times 10^{-25}$; length of actuator domain (μm), $\Delta = 5$; actuator gap (μm), $d_0 = 10$; permitivity of free space (farad/m), $\varepsilon_0 = 8{\cdot}85 \times 10^{-12}$.